FLOW OF A SONIC GAS STREAM PAST A BODY OF REVOLUTION

(OBTEKANIE TELA VRASHCHENIIA ZVUKOVYM POTOKOM GAZA)

PMM Vol.28, № 2, 1964, pp. 280-284

S.V. FAL'KOVICH and I.A. CHERNOV
(Saratov)

(Received December 10, 1963)

In paper [1] the self-similar problem of axisymmetric near-sonic flow far from an arbitrary body was solved by a numerical method. The authors advanced the hypothesis that this solution corresponds to an exponent of self-similitude of $^4/_7$. In the present work a particular family of self-similar solutions is found algebraically in the s, t plane for both two-dimensional and axisymmetric flow, and the corresponding exponent of self-similitude is determined. The given family includes the solution of Guderley and Yoshihara. It is shown theoretically that the exponent of self-similitude for that solution is equal to $^4/_7$.

1. The approximate equation for the perturbation velocity potential Φ for two-dimensional and axisymmetric near-sonic flow can be written in the form

$$-(\varkappa+1)\,\Phi_x\Phi_{xx}+\Phi_{yy}+\omega\,\frac{\Phi_y}{y}=0 \qquad (\Phi_x=u,\ \Phi_y=v) \tag{1.1}$$

Here u and v are the components of the perturbation velocity for a basic sonic stream in a rectangular coordinate system, and $\omega = 0$ for two-dimensional flow and $\omega = 1$ for axisymmetric flow.

We consider self-similar solutions of Equation (1.1) of the form

$$\Phi = y^{3n-2} f(\zeta) \qquad \left(\zeta=\frac{x}{(\varkappa+1)^{1/3}\,y^n}\right) \tag{1.2}$$

Here n is the exponent of self-similitude. For the determination of $f(\zeta)$ we obtain the ordinary differential equation

$$f''(n^2\zeta^2-f')-n(5n-5+\omega)\,\zeta f'+(3n-2)(3n-3+\omega)\,f=0 \tag{1.3}$$

We introduce Guderley's variables

$$s=f\zeta^{-3}, \qquad t=f'\zeta^{-2} \tag{1.4}$$

which lead to an equation of first order

$$\frac{ds}{dt} = \frac{(n^2 - t)(t - 3s)}{2t^2 + B_1 t - B_2 s} \qquad \left(\begin{array}{l} B_1 = 3n^2 - 5n + n\omega \\ B_2 = (3n - 2)(3n - 3 + \omega) \end{array} \right) \tag{1.5}$$

In the finite part of the plane Equation (1.5) has the three singular points [1]

$$A\,(s = 0, t = 0), \qquad C\left(\frac{n^3(5n - 5 + \omega)}{(3n - 2)(3n - 3 + \omega)},\ n^2\right), \qquad D\left(s = \frac{3 - \omega}{9},\ t = \frac{3 - \omega}{3}\right)$$

If an integral curve $t = t(s)$ of Equation (1.5) is determined, then ζ is found from Formula [1]

$$\ln C\zeta = \int \frac{ds}{t - 3s} \tag{1.6}$$

The velocity components u and v are found from Formulas

$$\begin{aligned} u &= \Phi_x = (\varkappa + 1)^{-1/3} y^{2n-2} f'(\zeta) = (\varkappa + 1)^{-1/3} y^{2n-2} \zeta^2 t \\ v &= \Phi_y = y^{3n-3}\,[(3n - 2) f - n\zeta f'] = y^{3n-3} \zeta^3\,[(3n - 2)s - nt] \end{aligned} \tag{1.7}$$

2. We shall seek a particular family of solutions of the equation of the form

$$s = a + bt \pm \sqrt{1 + ct}\,(d + gt) \tag{2.1}$$

Analyzing the results of [2] and [3], it is easy to see that for $n = 2$ there exist solutions of the form (2.1) in both the two-dimensional and axisymmetric cases, which represent analytic flow in a Laval nozzle.

We substitute the function (2.1) in Equation (1.5). Equating coefficients of like powers of t, we obtain the following system of algebraic equations:

$$-B_2acd - 2B_2ag - 2B_2bd + 6n^2 d = 0 \tag{2.2}$$

$$cd(B_1 + 6n^2) + (2B_1 + 6n^2)g - 3B_2 bcd - 4B_2 bg + 3B_2acg - 6d = 0 \tag{2.3}$$

$$(3B_1 + 6n^2)cg - 5B_2bcg - 2g - 4cd = 0 \tag{2.4}$$

$$-3B_2cg^2 - 2b + 2 = 0 \tag{2.5}$$

$$(2B_1 + 6n^2)\,b - 4B_2cgd - 2B_2b^2 - 2B_2g^2 - 2n^2 - 6a = 0 \tag{2.6}$$

$$-2B_2ab - B_2cd^2 - 2B_2gd + 6n^2a = 0 \tag{2.7}$$

Equations (2.2) to (2.7) are a system of six algebraic equations for the determination of the unknowns a, b, c, d, g and n. In the general case they are easily transformed to a system of three equations for three unknowns.

Using (2.6) we determine a

$$a = b\left(\tfrac{1}{3}B_1 + n^2\right) - \tfrac{2}{3}B_2\,cgd - \tfrac{1}{3}B_2b^2 - \tfrac{1}{3}\,B_2g^2 - \tfrac{1}{3}n^2 \tag{2.8}$$

We express b through (2.5).

$$b = 1 - \tfrac{3}{2}B_2\,cg^2 \tag{2.9}$$

From Equation (2.4) with the use of (2.9) we determine d

$$d = g\left[\tfrac{3}{4} B_1 + \tfrac{2}{3} n^2 - \tfrac{5}{4} B_2 - \tfrac{1}{2} c^{-1}\right] + \tfrac{15}{8} B_2^2 c g^3 \tag{2.10}$$

Substituting these values of a, b and d into Equations (2.2), (2.3) and (2.7), we obtain a system of three equations for determining c, g and n. However this system has a very unwieldy form and its immediate solution is difficult.

3. We consider first a solution of the form (2.1) passing through the singular point A, which is node of the integral curves of Equation (1.5) with angle of inclination

$$\frac{ds}{dt} = \frac{n}{3n-2} \tag{3.1}$$

It is known [1] that the singular point A represents in s, t variables the x-axis of the physical plane. Condition (3.1) on solutions of the type (2.1) indicates an important property of symmetry of the flow with respect to the x-axis in the case $\omega = 0$ and absence from that axis of singularities of source type in the case $\omega = 1$.

For such solutions it is possible to write the two additional conditions

$$d = -a, \qquad b = \frac{n}{3n-2} + \frac{ca}{2} - g \tag{3.2}$$

Substituting (3.2) into Equations (2.2) and (2.7) we obtain a system of four equations for the determination of a, c, g and n whose solution, under the conditions that $a \neq 0$ and $g \neq 0$, has the form

$$g = \frac{4n\omega - 12n \pm \sqrt{720n^3 - 1476n^2 + 720n - 48n^3\omega + 308n^2\omega - 272n\omega}}{12B_2}$$

$$a = \frac{60n^3 - 4n^3\omega - 138n^2 + {}^{94}/_3\, n^2\omega + 60n - {}^{68}/_3\, n\omega}{12B_2} \tag{3.3}$$

$$b = \frac{6n^2 - 3n + n\omega}{2B_2}, \qquad c = \frac{12n^2 - 27n + 5n\omega + 12 - 4\omega}{3B_2^2 g^2}$$

Finally it is possible to obtain an equation for the determination of the exponent of self-similitude n.

In the case of two-dimensional flow ($\omega = 0$) we will have

$$\pm\sqrt{20n^3 - 41n^2 + 20n}\,(2n^2 - 41n + 20) - 80n^4 + 304n^3 - 447n^2 + 304n - 80 = 0 \tag{3.4}$$

Freeing this of irrationality we obtain

$$\begin{aligned} &3200n^8 - 28320n^7 + 106568n^6 - 222638n^5 + 282381n^4 - \\ &- 222638n^3 + 106568n^2 - 28320n + 3200 = 0 \end{aligned} \tag{3.5}$$

We find the roots

$$n_1 = 2, \quad n_2 = n_3 = \frac{4}{5}, \quad n_4 = n_5 = \frac{5}{4}, \quad n_6 = \frac{1}{2}, \quad n_7 = \frac{9+\sqrt{17}}{8}, \quad n_8 = \frac{9-\sqrt{17}}{8}$$

The roots n_7 and n_8 should be discarded; they correspond to solutions with $g = 0$, which do not satisfy the basic system (2.2) to (2.7).

In the case of axisymmetric flow ($\omega = 1$), the equation determining the exponent of self-similitude has the form

$$\pm \sqrt{42n^3 - 73n^2 + 28n}\,(42n^2 - 73n + 28) - 252n^4 + 816n^3 - 993n^2 + 544n - 112 = 0 \quad (3.6)$$

Freeing this irrationality we obtain

$$\begin{gathered} 31752n^8 - 242676n^7 + 776322n^6 - 1357191n^5 + 1417205n^4 - \\ - 904794n^3 + 345032n^2 - 71904n + 6272 = 0 \end{gathered} \quad (3.7)$$

We find the roots

$$n_1 = 2, \quad n_2 = \frac{1}{3}, \quad n_3 = n_4 = \frac{4}{7}, \quad n_5 = \frac{4}{3}, \quad n_6 = n_7 = \frac{7}{6}, \quad n_8 = \frac{1}{2}$$

The roots n_5 and n_8 should be discarded.

4. Using the values found for n to find the coefficients a, b, c, d and g from the relations (3.3) and (3.7), we obtain the following system of solutions:

In the plane case ($\omega = 0$)

$$n = 2, \qquad s = \frac{1}{3} + \frac{3}{4}t \pm \sqrt{1+2t}\left(-\frac{1}{3} + \frac{1}{12}t\right), \qquad t > -\frac{1}{2} \quad (4.1)$$

$$n = \frac{4}{5}, \qquad s = \frac{10}{3} - 3t \pm \frac{10}{3}\sqrt{(1-t)^3}, \qquad t < 1 \quad (4.2)$$

$$n = \frac{5}{4}, \qquad s = -\frac{125}{84} + \frac{15}{7}t \pm \frac{25}{84}\sqrt{\left(1 - \frac{16}{25}t\right)^3}, \qquad t < \frac{25}{16} \quad (4.3)$$

$$n = \frac{1}{2}, \qquad s = \frac{1}{3} \pm \sqrt{1 + \frac{8}{3}t\left(-\frac{1}{2} + \frac{1}{3}t\right)}, \qquad t > -\frac{3}{8} \quad (4.4)$$

In the axisymmetric case ($\omega = 1$)

$$n = 2, \qquad s = \frac{1}{2} + \frac{5}{8}t \pm \sqrt{1+t}\left(-\frac{1}{2} + \frac{1}{8}t\right) \qquad t > -1 \quad (4.5)$$

$$n = \frac{1}{3}, \qquad s = \frac{2}{9} \pm \sqrt{1+6t}\left(-\frac{2}{9} + \frac{1}{3}t\right), \qquad t > -\frac{1}{6} \quad (4.6)$$

$$n = \frac{4}{7}, \qquad s = -\frac{28}{9} + 5t \pm \frac{28}{9}\sqrt{\left(1 - \frac{3}{2}t\right)^3}, \qquad t < \frac{2}{3} \quad (4.7)$$

$$n = \frac{7}{6}, \qquad s = -\frac{343}{729} + \frac{35}{27}t \pm \frac{343}{729}\sqrt{\left(1 - \frac{36}{49}t\right)^3}, \qquad t < \frac{49}{36} \quad (4.8)$$

Thus all solutions of the form (2.1) have been found that pass through the singular point A with angle of inclination (3.1).

Using the other singular points of Equation (1.5) and the known slopes of the integral curves at these points, it is possible to obtain the remaining solutions of the form (2.1).

5. It is noteworthy that the series of solutions obtained above include the following: flow in a plane Laval nozzle [5 and 2], solution (4.1); flow far from an arbitrary body in a two-dimensional near-sonic flow, first considered by Frankl' [4], solution (4.2), In fact, the assignment of a definite value of n and the condition that in the vicinity of the point A in the st plane the integral curve has an expansion of the form

$$s = \frac{n}{3n-2}\ t + A_2 t^2 + \ldots$$

uniquely determines these solutions.

Solution (4.7) satisfies all the conditions formulated in [1], and determines the flow far from a body in axisymmetric near-sonic flow. Thus the exponent of self-similitude corresponding to that flow is equal to ${}^4/_7$.

6. In order to find the potential $\Phi(x,y)$ of this flow, we use Equation (1.6). Substituting the value of s from Formula (4.7) into (1.6) we obtain

$$\ln C\zeta = \int \frac{(5 - 7\sqrt{1 - {}^3/_2 t})\, dt}{{}^{28}/_3 - 14t - {}^{28}/_3 \sqrt{(1 - {}^3/_2 t)^3}} \tag{6.1}$$

We set

$$z^2 = 1 - \frac{3}{2} t \tag{6.2}$$

After integration we find to within a scale constant

$$\zeta = -(1 - z)^{-2/7} z^{-5/7} \tag{6.3}$$

Substituting (6.3) into the first of Equations (1.4) we will have

$$f = -{}^1/_9 (1 - z)^{-6/7} z^{-15/7} [2 - 30z^2 + 28z^3] \tag{6.4}$$

Equations (6.3) and (6.4) parametrically determine the function $f = f(\zeta)$.

We now find the sonic line. Along it, according to Equation (1.7), $t = 0$. From (6.2) we have $z = -1$. Then according to Equation (6.3)

$$\zeta_* = \frac{1}{2^{2/7}}$$

The limiting characteristic corresponds to $t = n^2 = {}^{16}/_{49}$, and from the relation (6.3) we find

$$\zeta_C = \frac{7}{5^{5/7}\, 12^{2/7}}$$

Behind the limiting characteristic there exists a line on which the velocity is horizontal. For the determination of that line we find from Equation (1.7)

$$v = 0 \text{ for } (3n - 2)s - nt = 0$$

For $n = {}^4/_7$ this gives $s = -2t$. Substituting this value of s into (4.7) we find, using (6.2),

$$2z^3 - 3z^2 + 1 = 0$$

This equation has the roots $z = 1$, corresponding to the negative part of the x-axis, and $z = -\frac{1}{2}$. According to (6.3) the latter root gives

$$\zeta_{v=0} = \frac{2}{3^{2/7}}$$

According to (6.2) the point D $(t = {}^2/_3)$ corresponds to $z = 0$, which gives $\zeta = \infty$. Here we reach the positive part of the x-axis. It is evident from (1.7) that v is different from zero. Consequently this part of the x-axis is covered with sinks.

Analogous considerations apply also to the two-dimensional case [6].

BIBLIOGRAPHY

1. Guderley, K. and Yoshihara, H., An Axial-Symmetric Transonic Flow Pattern. Quart.Appl.Math. Vol.8, № 4, 1951. Russian transl. in the Sborn."Mekhanika", № 3, 1953.
2. Fal'kovich, S.V., K teorii sopla Lavalia (On the theory of the Laval nozzle). *PMM* Vol.10, № 4, 1946.
3. Ryzhov, O.S., O techeniiakh v okrestnosti perekhoda v soplakh Lavalia (On the flow in the transition region of a Laval nozzle). *PMM* Vol.22, № 4, 1958.
4. Frankl', F.I., Issledovanie po teorii kryla beskonechnogo razmakha, dvizhushchegosia so skorost'iu zvuka (Investigation of the theory of a wing of infinite span moving at sonic speed). Dokl.Akad.Nauk SSSR, Vol.58, № 7, 1947.
5. Frankl', F.I., K teorii sopel Lavalia (On the theory of the Laval nozzle). Izv.Akad.Nauk SSSR, ser.mat., Vol.9, № 5, 1945.
6. Fal'kovich, S.V. and Chernov, I.A., K teorii avtomodel'nykh okolozvukovykh techenii (On the theory of self-similar near-sonic flows). Izv. vyssh.ucheb.zaved., Mat., № 1, 1964.

Translated by M.D.V.D.